The final is comprehensive (8-9 pages).
There will be two pages on ch 9.

Ch. 9: For the final be able to

1. Solve separable diff. eq.
2. Use initial conditions & constants.
3. Set up and do ALL the applied problems from homework.

Worried about applied problems?
Pay attention today and Monday in lectures. Know the homework well. And go thru my review sheets and look at old finals.

Newton's Cooling Law Experiment
Hot water is in the cup. We will try to predict the temp. at the end of class.

1st measurement:
Time = 10:32 Temp = 160°F

2nd measurement:
Time = 10:45 Temp = 143°F

We'll use this later today

9.4 Differential Equations Applications

1. *Law of Natural Growth/Decay*:

Assumption: *"The rate of growth/decay is proportional to the function value."*

$$\frac{dP}{dt} = kP \text{ with } P(0) = P_0$$

We solve this in general last time and got

$$P(t) = P_0 e^{kt}$$

Example:

A population has 500 bacteria at t=0.
After 3 hrs there are 8000 bacteria.
Assume the pop. grows at a rate proportional to its size.
Find B(t).

$$\frac{dB}{dt} = kB$$

$B(0) = 500$

$B(3) = 8000$

$$\Rightarrow B(t) = B_0 e^{kt}$$

$B(0) = 500 \Rightarrow B_0 e^0 = 500 \Rightarrow B_0 = 500$

$B(3) = 8000 \Rightarrow 500 e^{3k} = 8000$

$\Rightarrow e^{3k} = 16$

$\Rightarrow 3k = \ln(16)$

$\Rightarrow k = \frac{1}{3}\ln(16) \approx 0.9242$

(RELATIVE GROWTH RATE = 92.42% per hr)

$$\boxed{B(t) = 500\, e^{\frac{1}{3}\ln(16)\, t}}$$

Example:

The *half-life* of cesium-137 is 30 years. Suppose we start with a 100-mg sample. Find m(t).

$$\frac{dm}{dt} = km$$

$$m(t) = m_0 e^{kt}$$

$$m(0) = 100 \Rightarrow m_0 = 100$$

HALF

$$m(30) = 50 \Rightarrow 100e^{30k} = 50$$

$$\Rightarrow e^{30k} = \tfrac{1}{2}$$

$$\Rightarrow 30k = \ln(\tfrac{1}{2})$$

$$\Rightarrow k = \frac{1}{30}\ln\left(\frac{1}{2}\right) \approx -0.0231$$

(relative growth rate = − 2.31% per year)

$$\boxed{m(t) = 100\, e^{\frac{1}{30}\ln\left(\frac{1}{2}\right)t}}$$

Example:

Bob deposits \$2000 into a savings account. The money grows at a rate proportional to its size (*i.e.* compound interest). The balance in 4 years is \$2100. Find the formula A(t) for the amount in his account in t years.

Pe^{rt}

$$\frac{dA}{dt} = rA \Rightarrow A(t) = A_0 e^{rt}$$

$$A(0) = 2000 \Rightarrow A_0 = 2000$$

$$A(4) = 2100 \Rightarrow 2000e^{4r} = 2100$$

$$\Rightarrow e^{4r} = \frac{2100}{2000} = 1.05$$

$$\Rightarrow 4r = \ln(1.05)$$

$$\Rightarrow r = \frac{1}{4}\ln(1.05)$$

$$\approx 0.012197$$

(1.22% annual rate)

$$\boxed{A(t) = 2000\, e^{\frac{1}{4}\ln(1.05)t}}$$

2. Newton's Law of Cooling:

Assumption: *"The rate of temperature change is proportional to the difference between the temperature of the object and its surroundings."*

$T(t)$ = temp of object

T_s = temp of surroundings = 70°F

Difference = $T - 70$

$$\frac{dT}{dt} = k(T-70)$$

$$\int \frac{1}{T-70}\,dT = \int k\,dt$$

$$\ln|T-70| = kt + C_1$$

$$|T-70| = e^{(kt+c_1)}$$

$$T-70 = \pm e^{c_1} e^{kt} \qquad c_2 = \pm e^{c_1}$$

$$T(t) = 70 + C_2 e^{kt}$$

General Sol'n

$$\boxed{T(t) = 70 + Ce^{kt}}$$

10:32 : $T(0) = 160$

$t=0$

$\Rightarrow 70 + Ce^0 = 160$

$\Rightarrow \boxed{C = 90}$

10:45 : $T(13) = 143$

$t=13$

$\Rightarrow 70 + 90e^{13k} = 143$

$\Rightarrow 90e^{13k} = 73$

$\Rightarrow e^{13k} = \frac{73}{90}$

$\Rightarrow 13k = \ln(73/90)$

$\Rightarrow k = \frac{1}{13}\ln(73/90) \approx -0.016103$

$$\boxed{T(t) = 70 + 90e^{-0.016t}}$$

At end of class, 11:30 $(t=48)$

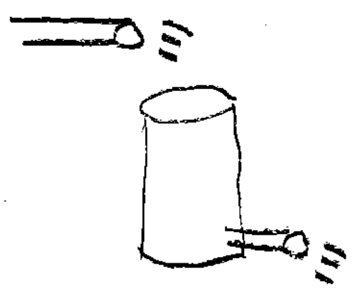
$T(48) = 70 + 90e^{-0.016(48)}$

$\approx \boxed{111.5496\ °F}$

3. *Mixing Problems:*

Assume you have a vat of liquid that has a substance (a contaminant) entering at some rate and exiting at some rate, then

"The rate of change of the contaminant is equal to the rate at which the contaminant is coming IN minus the rate at which it is going OUT."



These problems typically look like:

V	= volume of the vat	(liters)
t	= time	(min)
$y(t)$	= amount in vat	(kg)
$\dfrac{dy}{dt}$	= rate	(kg/min)

Thus,

$$\frac{dy}{dt} = \text{Rate In} - \text{Rate out}$$

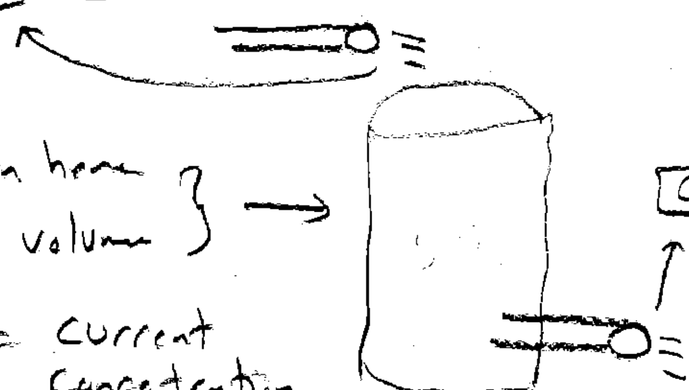
$$= \underbrace{\left(?\ \frac{kg}{L}\right)\left(?\ \frac{L}{min}\right)}_{\boxed{\text{IN}}} - \underbrace{\left(\frac{y}{V}\ \frac{kg}{L}\right)\left(?\ \frac{L}{min}\right)}_{\boxed{\text{OUT}}}$$

$$y(0) = ?\ \text{kg}$$

$y(t)$ = kg in here

V = total volume

$\dfrac{y(t)}{V}$ = current concentration



Example:

Assume a 10 Liter vat contains 2kg of salt initially. A pipe pumps in salt water (brine) at 5 L/min with a concentration of 3 kg/L of salt.

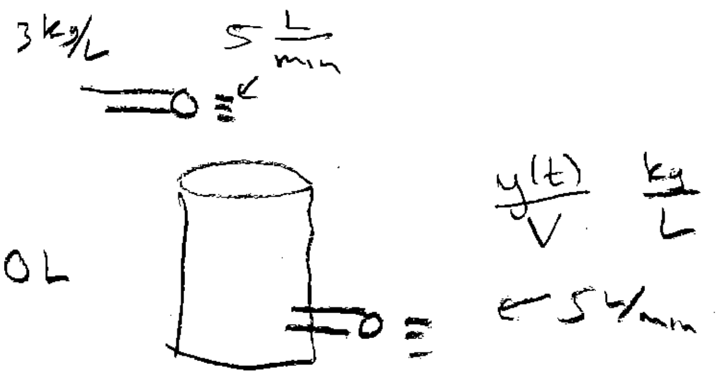
The vat is well mixed.

The mixture leaves the vat at 5L/min.

Let $y(t)$ = the amount of salt in the vat at time t.

(a) Find $y(t)$.

(b) Find the limit of $y(t)$ as $n \to \infty$.



$y(t)$ = kg of salt in vat.

$$\frac{dy}{dt} = \underbrace{15\ \frac{kg}{min}}_{3\cdot 5} - \frac{y}{10}\cdot 5\ \frac{kg}{min}$$

$$\uparrow \frac{y}{V}$$

$$\boxed{\frac{dy}{dt} = 15 - \frac{1}{2}y \qquad y(0) = 2}$$

$$\frac{1}{15-\frac{1}{2}y}\,dy = 1\,dt$$

$$\int \frac{2}{30-y}\,dy = \int dt$$

$$-2\ln|30-y| = t + C_1$$

$$\ln|30-y| = -\tfrac{1}{2}t - \tfrac{1}{2}C_1$$

$$|30-y| = e^{(-\frac{1}{2}t - \frac{1}{2}C_1)}$$

$$30 - y = \pm e^{-\frac{1}{2}C_1} e^{-\frac{1}{2}t} \qquad C_2 = \pm e^{-\frac{1}{2}C_1}$$

$$\boxed{y = 30 - C_2 e^{-\frac{1}{2}t}}$$

$$y(0) = 2 \Rightarrow 2 = 30 - C \Rightarrow C = 28$$

$$\boxed{y(t) = 30 - 28e^{-\frac{1}{2}t}} \qquad \boxed{\lim_{t\to\infty} y(t) = 30\ kg}$$

Example:

Assume a 100 Liter vat contains 5kg of salt initially. Two pipes (A & B) pump in salt water (brine).

Pipe A: Enters at 3L/min with a concentration of 4kg/L of salt.

Pipe B: Enters at 5L/min with a concentration of 2kg/L of salt.

The vat is well mixed.

The mixture leaves the vat at 8L/min.

Let $y(t)$ = the amount of salt in the vat at time t.

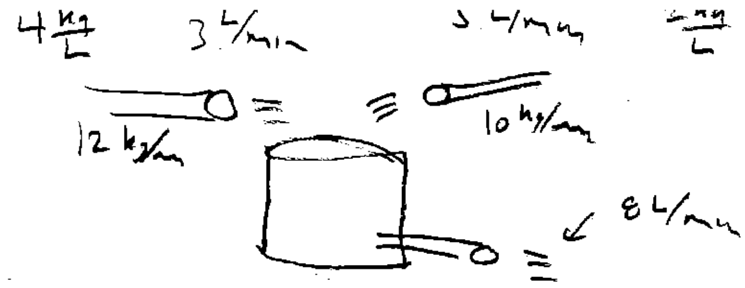
(c) Find $y(t)$.

(d) Find the limit of $y(t)$ as $n \to \infty$.

$y(t) = 275 + Ce^{-2/25\, t}$

$y(0) = 5 \Rightarrow C = -270$

$\lim_{t \to \infty} y(t) = 275$



$y(t)$ = kg of salt in vat

$$\frac{dy}{dt} = 22 - \frac{y}{100} \cdot 8$$

$$y(0) = 5$$

$$\boxed{\frac{dy}{dt} = 22 - \frac{2}{25} y \qquad y(0) = 5}$$

$$\int \frac{1}{22 - \frac{2}{25} y}\, dy = \int 1\, dt$$

$$-\frac{25}{2} \ln\left|22 - \frac{2}{25} y\right| = t + C_1 \qquad C_2 = -\frac{2}{25} C_1$$

$$\ln\left|22 - \frac{2}{25} y\right| = -\frac{2}{25} t + C_2$$

$$22 - \frac{2}{25} y = \pm e^{C_2} e^{-\frac{2}{25} t} \qquad C_3 = \pm e^{C_2}$$

$$\frac{2}{25} y = 22 - C_3 e^{-\frac{2}{25} t}$$

$$y = \frac{25}{2} \cdot 22 - \frac{25}{2} C_3 e^{-\frac{2}{25} t}$$